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EXISTENCE OF NASH EQUILIBRIA ON INTEGER PROGRAMMING GAMES

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Existence of Nash equilibria on integer programming games

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Abstract

We aim to investigate a new class of games, where each player's set of strategies is a union of polyhedra. These are called *integer programming games.* We prove that it is a Σ_2^p-complete problem to decide the existence of *Nash equilibria* and we provide sufficient conditions for an equilibrium to exist. Additionally, we describe examples suitable to be modeled by an integer programming game.

1 Introduction

Game Theory. Game theory is a generalization of decision theory. In a game, each player is a decision maker that aims to maximize her utility, which is influenced by other participants' decisions. See Fudenberg and Tirole [9] and Owen [20] for an introduction to this field.

We restrict our investigation to *non-cooperative* games, *i.e.*, players have no compassion for the opponents. Nash [18] defined the most widely accepted concept of solution for a game, the *Nash equilibrium* (NE). A NE associates a probability distribution to each player set of strategies such that no player has incentive to unilaterally deviate from that NE if the others play according with the equilibrium. In other words, in an equilibrium, each player is maximizing her expected utility given the equilibrium strategies of the other players. In a *pure* NE only a single strategy of each player has positive probability assigned (*i.e.*, probability 1).

The state-of-the-art game theory tools are confine to finite games, where each player has a finite number of strategies and all players' strategies combinations are explicitly enumerated with the associated utilities, and "well-behaved" continuous games, where utility functions and strategy sets meet certain differentiability and concavity conditions.

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Finite Games. These games are represented in *normal-form* (or strategic-form), *i.e.*, through a multidimensional matrix, where each entry corresponds to the players' utilities for a given combination of their strategies. Nash [18] proved that any finite game has an NE.

Finite games have received wide attention in game theory. Daskalis *et al.* [8] prove that computing an NE is PPAD-complete, which is believed to be a class of hard problems, since it is unlikely that $PPAD$ is equal to the polynomial time complexity class P. Nisan *et al.* [19] describe general algorithms to compute Nash equilibria, which failed to run in polynomial time. We refer the interested reader to the surveys and state-of-art algorithms collected in von Stengel [25]. Currently, some of those algorithms are available on GAMBIT [17], the most up-to-date software for the computation of NE for normal-form games.

Continuous games. The class of continuous games considers broader strategy sets with respect to finite games, allowing the sets of strategies to be infinite; see Figure 1. Apart from finite games, the literature focuses in continuous games for which the strategies sets are convex and utilities are concave. Examples of such games are the well-studied economic models of Cournot [7] and Bertrand [2] competitions. In these games, the players' are firms producing a homogeneous good. In the Cournot competition, the firms compete on the quantities to be placed in the market while in the Bertrand model, the firms compete on prices. The firms' decisions will influence the overall demand. In the classical versions of these models, the players' maximization problems are concave which enables the application of the famous theorem by Debreu, Glicksberg and Fan (see [9] Chapter 1) stating the existence of *pure* NE. Moreover, the computation of a pure NE can be reduced to a constrained problem by the application of the Karush-Kuhn-Tucker [12, 15] (KKT) conditions to each player's optimization problem.

Separable games [24] are continuous games in which each player's utility function has a particular form that, in particular, allows to formulate finite games; see Figure 1.

Integer programming games. Let $M = \{1, 2 \ldots, m\}$ be a set of players. Based on the definition presented by Köppe *et al.* [13], we define an *integer programming game* (IPG) as a non-cooperative game, where each player p's goal is to select her *best reaction* x^p against the opponents' strategies, $x^{-p} = (x^1, \ldots, x^{p-1}, x^{p+1}, \ldots, x^m)$, by solving the following mathematical programming problem:

$$\begin{aligned} \underset{x^p \geq 0}{\text{maximize}} \quad & \Pi^p(x^p, x^{-p}) && \text{(1a)} \\ \text{subject to} \quad & A^p x^p \leq b^p && \text{(1b)} \\ & x_i^p \in \mathbb{N} \text{ for } i = 1, \ldots, B_p, && \text{(1c)} \end{aligned}$$

where A^p is a $k_p \times n_p$ matrix (with $n_p \geq B_p$), b^p a column vector of dimension k_p and $\Pi^p(x^p, x^{-p})$ is player p's utility function.

Note that IPGs contain mathematical programming problems in the special case of a single player. Moreover, any finite game can be modeled through an IPG: associate a binary variable for each player pure strategy (which would model the strategy selected), add a constraint summing the decision variables

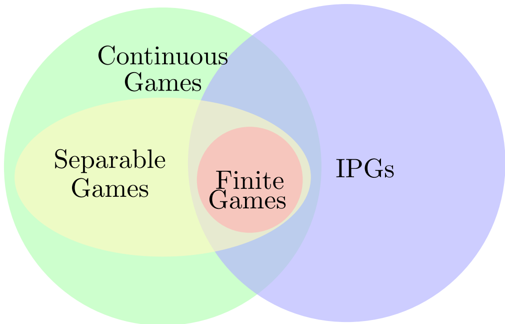

Figure 1: Games classes.

up to one (this ensures that exactly one strategy is selected) and formulate the players' utilities according to the utility values for combinations of the binary variables; see Figure 1. On the other hand, enumerating all players' feasible strategies (as in finite games) for an IPG can be impractical, or the players' strategies in an IPG might lead to non well-behaved games, for example where the player's maximization problems are non-concave. This shows that the existent tools and standard approaches for finite games and convex games are not directly applicable to IPGs.

The literature in IPGs is scarce and often focus in the particular structure of specific games. Kostreva [14] and Gabriel *et al.* [10] propose methods to compute NE for IPGs, however it lacks a computational time complexity guarantee and a practical validation through computational results. Köppe *et al.* [13] present a polynomial time algorithm to compute pure NE (under restrictive conditions, like number of players fixed and sum of the number of player's decision variables fixed, to name few).

There are important real-world IPGs, in the context of *e.g.*, electricity markets [22], production planning [16], heath-care [5]); this highlights the importance of exploring such game models.

Our contributions and paper organization. We highlight three contributions concerning IPGs: the computational complexity study of the problem of deciding the existence of a pure NE and of a NE, and the determination of sufficient conditions to guarantee the existence of NE.

Our paper is structured as follows. Section 2 fixes notation and covers the game theory background. Section 3 classifies the computational complexity of the problems related with the existence of NE to IPGs and states sufficient conditions for NE to exist. Section 4 presents examples of integer programming games. Finally, we conclude and discuss further research directions in Section 5.

2 Notation and background

Notation. The feasible set of strategies X^p is

$$X^p = \{x^p : A^p x^p \leq b^p, x_i^p \in \mathbb{N} \text{ for } i = 1, \dots, B_p\}.$$

Let the operator $(\cdot)^{-p}$ denote $(\cdot)$ for all players except player p. We denote the set of all players' strategies combinations by X, *i.e.*, $X = \prod_{p \in M} X^p$. We call each $x^p \in X^p$ and $x \in X$ a player p pure strategy and a pure profile of strategies, respectively.

We assume that there is *complete information*, *i.e.*, players have full information about each other utilities and strategies, players select their strategies simultaneously and each player utility is a continuous function of $x \in X$ and can be evaluated in polynomial time. We assume that all players are rational, and thus, each player p goal is to select her *best reaction* to x^{-p} by solving problem (1).

Game theory background. A pure profile of strategies $x \in X$ that solves the optimization problem (1) for all players is called *pure equilibrium*. A game may fail to have pure equilibria and, therefore, a broader solution concept for a game must be introduced. To that end, we introduce some basic concepts of measure theory. Let Δ^p denote the space of Borel probability measures over X^p and $\Delta = \prod_{p \in M} \Delta^p$. Each player p expected utility for a profile of strategies $\sigma \in \Delta$ is

$$\Pi^p(\sigma) = \int_{X^p} \Pi^p(x^p, x^{-p}) d\sigma. \tag{2}$$

A *Nash equilibrium* (NE) is a profile of strategies $\sigma \in \Delta$ such that

$$\Pi^p(\sigma) \geq \Pi^p(x^p, \sigma^{-p}), \qquad \forall p \in M \qquad \forall x^p \in X^p. \tag{3}$$

In an NE each player p's expected profit from σ cannot be improved by unilaterally deviating to a different strategy[1].

The *support of a strategy* $\sigma^p \in \Delta^p$, denoted as $\text{supp}(\sigma^p)$, is the set of player p's strategies played with positive probability, *i.e.*,

$$\text{supp}(\sigma^p) = \{x^p \in X^p : \quad \sigma^p(x^p) > 0\}.$$

Given $\sigma \in \Delta$, if each player support size is 1, then it is a pure profile of strategies, otherwise, we call it (strictly) mixed. For the sake of simplicity, whenever the context makes it clear, we use the term (strategy) profile to refer to a pure one.

A game is called *continuous* if each player p strategy set is a nonempty compact metric space and the utility is continuous.

A *separable game* is a continuous game with utility functions taking the form

$$\Pi^p(x^p, x^{-p}) = \sum_{j_1=1}^{k_1} \cdots \sum_{j_m=1}^{k_m} a^p_{j_1 \ldots j_m} f^1_{j_1}(x^1) \ldots f^m_{j_m}(x^m). \tag{4}$$

where $a^p_{j_1 \ldots j_m} \in \mathbb{R}$ and the f^p_j are real-valued continuous functions.

[1]The equilibrium conditions (3) only reflect a player p deviation to strategy in X^p and not in Δ^p, because a strategy in Δ^p is a convex combination of strategies in X^p, and thus cannot lead to a better utility than one in X^p.

3 Existence of Nash equilibria

It can be argued that players' computational power is bounded and thus, since the space of pure strategies is simpler and contained in the space of mixed strategies – *i.e.*, the space of Borel probability measures – pure equilibria are more plausible outcomes for games with large sets of pure strategies. In this way, it is important to understand the complexity of determining a pure equilibrium to an IPG.

According with Nash famous theorem [18] any finite game has a Nash equilibrium. Since a purely integer bounded IPGs is a finite game, it has an NE. However, Nash theorem does not guarantee that the equilibrium is pure, which is illustrated in the following example.

Example 3.1 (No pure Nash equilibrium.) *Consider the duopoly game such that player A solves*

$$\begin{aligned} \underset{x^A}{\text{maximize}} \quad & 18x^A x^B - 9x^A \\ \textit{subject to} \quad & x^A \in \{0,1\} \end{aligned}$$

and player B:

$$\begin{aligned} \underset{x^B}{\text{maximize}} \quad & -18x^A x^B + 9x^B \\ \textit{subject to} \quad & x^B \in \{0,1\}. \end{aligned}$$

Under the outcome $(x^A, x^B) = (0,0)$ *player B has incentive to change to* $x^B = 1$*; for the outcome* $(x^A, x^B) = (1,0)$ *player A has incentive to change to* $x^A = 0$*; for the outcome* $(x^A, x^B) = (0,1)$ *player A has incentive to change to* $x^A = 1$*; for the outcome* $(x^A, x^B) = (1,1)$ *player B has incentive to change to* $x^B = 0$*. Thus there is no pure NE.*

In Section 3.1, we classify both the computational complexity of deciding if there is a pure and a mixed NE for an IPG. It will be shown that even with linear utilities and two players, the problem is Σ_2^p-complete. Then, in Section 3.2, we state sufficient conditions for the game to have finitely supported Nash equilibria.

3.1 Complexity of the existence of NE

The complexity class Σ_2^p contains all decision problems that can be written in the form $\exists x \forall y P(x,y)$, that is, as a logical formula starting with an existential quantifier followed by a universal quantifier followed by a Boolean predicate $P(x,y)$ that can be evaluated in polynomial time; see Chapter 17 in Papadimitriou's book [21].

Theorem 3.2 *The problem of deciding if an IPG has a pure NE is* Σ_2^p*-complete.*

Proof. The decision problem is in Σ_2^p, since we are questing if there is a solution in the space of pure strategies such that for any unilateral deviation of a player, her utility is not improved (and evaluating the utility value for a profile of strategies can be done in polynomial time).

It remains to prove Σ_2^p-hardness. We will reduce the following Σ_2^p-complete to it (see Caprara *et al.* [3]):

DeNegre bilevel Knapsack Problem - DN

INSTANCE Non-negative integers n, A, B, and n-dimensional non-negative integer vectors a and b.

QUESTION Is there a binary vector x such that $\sum_{i=1}^n a_i x_i \leq A$ and for all binary vectors y with $\sum_{i=1}^n b_i y_i \leq B$, the following inequality is satisfied

$$\sum_{i=1}^n b_i y_i (1 - x_i) \leq B - 1?$$

Our reduction starts from an instance of DN. We construct the following instance of IPG.

- The game has two players, $M = \{Z, W\}$.
- Player Z controls a binary decision vector z of dimension $2n + 1$; her set of feasible strategies is

$$\begin{aligned} \sum_{i=1}^n a_i z_i &\leq A \\ z_i + z_{i+n} \leq & \quad 1 \qquad i = 1, \dots, n \\ z_{2n+1} + z_{i+n} \leq & \quad 1 \qquad i = 1, \dots, n. \end{aligned}$$

- Player W controls a binary decision vector w of dimension $n + 1$; her set of feasible strategies is

$$Bw_{n+1} + \sum_{i=1}^n b_i w_i \leq B. \tag{8}$$

- Player Z's utility is $(B - 1)w_{n+1} z_{2n+1} + \sum_{i=1}^n b_i w_i z_{i+n}$.
- Player W's utility is $(B-1)w_{n+1} + \sum_{i=1}^n b_i w_i - \sum_{i=1}^n b_i w_i z_i - \sum_{i=1}^n b_i w_i z_{i+n}$.

We claim that in the constructed instance of IPG there is an equilibrium if and only if the DN instance has answer YES.

(Proof of if). Assume that the DN instance has answer YES. Then, there is x satisfying $\sum_{i=1}^n a_i x_i \leq A$ such that $\sum_{i=1}^n b_i y_i (1 - x_i) \leq B - 1$. Choose as strategy for player Z, $\widehat{z} = (x, \overbrace{0, \dots, 0}^{n}, 1)$ and for player W $\widehat{w} = (\overbrace{0, \dots, 0}^{n}, 1)$. We will prove that $(\widehat{z}, \widehat{w})$ is an equilibrium. First, note that these strategies are guaranteed to be feasible for both players. Second, note that none of the players has incentive to deviate from $(\widehat{z}, \widehat{w})$:

- Player Z's utility is $B - 1$, and $B - 1 \geq \sum_{i=1}^n b_i w_i$ holds for all the remaining feasible strategies w of player W.

- Player W's has utility $B-1$ which is the maximum possible given $\widehat{z}$.

(Proof of only if). Now assume that the IPG instance has answer YES. Then, there is a pure equilibrium $(\widehat{z}, \widehat{w})$.

If $\widehat{w}_{n+1} = 1$, then, by (8), $\widehat{w} = (\overbrace{0, \dots, 0}^{n}, 1)$. In this way, since player Z maximizes her utility in an equilibrium, $\widehat{z}_{2n+1} = 1$, forcing $\widehat{z}_{i+n} = 0$ for $i = 1, \dots, n$. The equilibrium inequalities (3), applied to player W, imply that, for any of her feasible strategies w with $w_{n+1} = 0$,

$$B - 1 \geq \sum_{i=1}^{n} b_i w_i (1 - \widehat{z}_i)$$

holds, which shows that DN is a YES instance with the leader selecting $x_i = \widehat{z}_i$ for $i = 1, \dots, n$.

If $\widehat{w}_{n+1} = 0$, under the equilibrium strategies, player Z's utility term $(B - 1)\widehat{w}_{n+1} z_{2n+1}$ is zero. Thus, since in an equilibrium player Z maximizes her utility, it holds that $\widehat{z}_{i+n} = 1$ for all $i = 1, \dots, n$ with $\widehat{w}_i = 1$. However, this implies that player W's utility is non-positive given the profile $(\widehat{z}, \widehat{w})$. In this way, player W would strictly improve her utility by unilaterally deviating to $w = (\overbrace{0, \dots, 0}, 1)$. In conclusion, w_{n+1} is never zero in a pure equilibrium of the constructed game instance. $\square$

Extending the existence property to mixed equilibria would increase the chance of an IPG to have an NE, and thus, a solution. The next theorem shows that the problem remains Σ_2^p-complete.

Theorem 3.3 *The problem of deciding if an IPG has an NE is Σ_2^p-complete.*

Proof. Analogously to the previous proof, the problem belongs to Σ_2^p.

It remains to show that it is Σ_2^p-hard. We will reduce the following Σ_2^p-complete to it (see [3]):

Dempe Ritcht Problem - DR

INSTANCE Non-negative integers n, A, C and C', and n-dimensional non-negative integer vectors a and b.

QUESTION Is there a value for x such that $C \leq x \leq C'$ and for all binary vectors satisfying $\sum_{i=1}^{n} b_i y_i \leq x$, the following inequality holds

$$Ax + \sum_{i=1}^{n} a_i y_i \geq 1?$$

Our reduction starts from an instance of DR. We construct the following instance of IPG.

- The game has two players, $M = \{Z, W\}$.

- Player Z controls a non-negative variable z and a binary decision vector $(z_1, \dots, z_{n+1})$; her set of feasible strategies is

$$\begin{aligned} \sum_{i=1}^{n} b_i z_i \leq & \ z \\ z_i + z_{n+1} \leq 1, & \qquad\qquad i = 1, \dots, n \\ z \leq & C'(1 - z_{n+1}) \\ z \geq & \ C(1 - z_{n+1}). \end{aligned}$$

- Player W controls a non-negative variable w and binary decision vector $(w_1, \dots, w_n)$.
- Player Z's utility is $Az + \sum_{i=1}^{n} a_i z_i w_i + z_{n+1}$.
- Player W's utility is $z_{n+1} w + \sum_{i=1}^{n} b_i w_i z_i$.

We claim that in the constructed instance of IPG there is an equilibrium if and only if the DR instance has answer YES.

(Proof of if). Assume that the DR instance has answer YES. Then, there is x such that $C \leq x \leq C'$ and $Ax + \sum_{i=1}^{n} a_i y_i \geq 1$ for a y satisfying $\sum_{i=1}^{n} b_i y_i \leq x$. As strategy for player Z choose $\widehat{z} = C'$ and $(\widehat{z}_1, \dots, \widehat{z}_n, \widehat{z}_{n+1}) = (y_1, \dots, y_n, 0)$; for player W choose $\widehat{w} = 0$ and $(\widehat{w}_1, \dots, \widehat{w}_n) = (y_1, \dots, y_n)$. We prove that $(\widehat{z}, \widehat{w})$ is an equilibrium. First, note that these strategies are guaranteed to be feasible for both players. Second, note that none of the players has incentive to deviate from $(\widehat{z}, \widehat{w})$:

- Player Z's utility cannot be increased, since it is equal or greater than 1 and for $i = 1, \dots, n$ such that $\widehat{z}_i = 0$ the utility coefficients are zero.
- Analogously, player W's utility cannot be increased, since for $i = 1, \dots, n$ such that $\widehat{w}_i = 0$ the utility coefficients are zero and the utility coefficient of $\widehat{z}_{n+1} \widehat{w}$ is also zero.

(Proof of only if). Assume that DR is a NO instance. Then, for any x in $[C, C']$ the leader is not able to guarantee a utility of 1. This means that in the associated IPG, player Z has incentive to choose $z = 0$ and $(z_1, \dots, z_n, z_{n+1}) = (0, \dots, 0, 1)$. However, this player Z's strategy leads to a player W's unbounded utility. In conclusion, there is no equilibrium. $\square$

In the proof of Theorem 3.3, it is not used the existence of a mixed equilibrium to the constructed IPG instance. Therefore, it implies Theorem 3.2. The reason for presenting these two theorems is because in Theorem 3.2, the reduction is a game where the players have finite sets of strategies, while in Theorem 3.3, in the reduction, a player has an unbounded set of strategies.

3.2 Conditions for the existence of NE

Glicksberg [11] and Stein et al. [24] provide results on the existence and characterization of equilibria for continuous and separable games (recall the definitions in Section 2), which we will apply to IPGs. In an IPG, each player p's strategy set X^p is a nonempty compact metric space if X^p is bounded and nonempty. This together with the fact that in Section 2 we assumed that each player utility is continuous, allow us to conclude the following (explaining Figure 1):

Lemma 3.4 *Every IPG such that X^p is nonempty and bounded is a continuous game.*

Glicksberg [11] proved that every continuous game has a NE. Thus,

Theorem 3.5 *Every IPG such that X^p is nonempty and bounded has a Nash equilibrium.*

Applying Stein et al. [24] results, we obtain the following:

Theorem 3.6 *For any Nash equilibrium σ of a separable IPG, there is a Nash equilibrium τ such that each player p mixes among at most k_p+1 pure strategies and $\Pi^p(\sigma) = \Pi^p(\tau)$.*

Proof. Apply Theorem 2.8 of [24] to a separable IPG. □

If in an IPG each player set of strategies X^p is bounded and the utility takes the form (4), IPG is separable. Assuming that these two conditions are satisfied (so that Theorem 3.5 and Theorem 3.6 hold) is not too strong when modeling real-world applications. In other words, the players' strategies are likely to be bounded due to limitations in the players' resources, which guarantees that an IPG has an equilibrium (Theorem 3.5). Moreover, interesting IPGs, as the models that we present later in Section **??**, possess quadratic utility functions that can be written in the form (4).

Corollary 3.7 *Let IPG be such that X^p is nonempty and bounded, and*

$$\Pi^p(x^p, x^{-p}) = c^p x^p + \sum_{k\in M} (x^k)^\intercal Q^p_k x^p, \tag{10}$$

where $c^p \in \mathbb{R}^{n_p}$ and Q^p_k is an n_k-by-n_p real matrix. Then, for any Nash equilibrium σ there is a Nash equilibrium τ such that each player p mixes among at most $1 + n_p + \frac{n_p(n_p-1)}{2}$ pure strategies and $\Pi^p(\sigma) = \Pi^p(\tau)$.

Proof. In order to write player p's utility in the form (4), there must be a function $f^p_{j_p}(x^p)$ for 1, x^p_1, …, $x^p_{n_p}$, $x^p_1x^p_1$, $x^p_1x^p_2$, …, $x^p_1x^p_{n_p}$, $x^p_2x^p_2$, …, $x^p_{n_p}x^p_{n_p}$; thus, $k_p = 1 + n_p + \frac{n_p(n_p-1)}{2}$ in Theorem 3.6.

The thesis [4] presents an algorithmic approach that uses the fact that we can restrict our investigations to finitely supported NE.

4 Examples

Next, we describe three games: the *knapsack game* which is the simplest purely integer programming game that one could devise, the *competitive lot-sizing game* and the *kidney exchange game* which have practical applicability in production planning and health-care, respectively.

4.1 Knapsack game.

One of the most simple and natural IPGs would be one with each player's utility function linear. Under this setting, each player p aims to solve

$$\max_{x^p \in \{0,1\}^n} \sum_{i=1}^{n} v_i^p x_i^p + \sum_{k=1, k\neq p}^{m} \sum_{i=1}^{n} c_{k,i}^p x_i^p x_i^k \tag{11a}$$

$$\text{s. t.} \quad \sum_{i=1}^{n} w_i^p x_i^p \leq W^p. \tag{11b}$$

The parameters of this game are integer (but are not required to be non-negative). This model can describe situations where m entities aim to decide in which of n projects to invest such that each entity budget constraint (11b) is satisfied and the associated utilities are maximized (11a).

In the knapsack game, each player p's set of strategies X^p is bounded, since she has at most 2^n feasible strategies. Therefore, by Corollary 3.6, it suffices to study finitely supported equilibria. Since utilities are linear, through the proof of Corollary 3.6, we deduce that the bound on the equilibria supports for each player is $n + 1$.

In Carvalho [4] mathematical programming tools are used to compute some refined equilibria of this game.

4.2 Competitive lot-sizing game.

The competitive lot-sizing game is a Cournot competition played through T periods by a set of firms (players) that produce the same good; see [4] for details. Each player has to plan her production as in the lot-sizing problems (see [23]) but, instead of satisfying a known demand in each period of the time horizon, the demand depends on the total quantity of the produced good that exists in the market. Each player p has to decide how much will be produced in each time period t (production variable x_t^p) and how much will be placed in the market (variable q_t^p). There are setup and variable (linear) production costs, upper limit on production quantities, and a producer can build inventory (variable h_t^p) by producing in advance. In this way, we obtain the following model for each player (producer) p:

$$\max_{y^p \in \{0,1\}^T, x^p, q^p, h^p} \sum_{t=1}^{T} P_t(q_t) q_t^p - \sum_{t=1}^{T} F_t^p y_t^p - \sum_{t=1}^{T} C_t^p x_t^p - \sum_{t=1}^{T} H_t^p h_t^p \tag{12a}$$

$$\text{s. t.} \quad x_t^p + h_{t-1}^p = h_t^p + q_t^p \qquad \text{for } t = 1, \ldots, T \tag{12b}$$

$$0 \leq x_t^p \leq M_t^p y_t^p \qquad \text{for } t = 1, \ldots, T \tag{12c}$$

where F_t^p is the setup cost, C_t^p is the variable cost, H_t^p is the inventory cost and M_t^p is the production capacity for period t; $P_t(q_t) = a_t - b_t \sum_{j=1}^{m} q_t^j$ is the unit market price. The utility function (12a) is player p's total profit; constraints (12b) model product conservation between periods; constraints (12c) ensure that the quantities produced are non-negative and whenever there is production ($x_t^p > 0$), the binary variable y_t^p is set to 1 implying the payment of the setup cost F_t^p.

4.3 Kidney exchange game

In order to increase the possibilities of a kidney patient being transplanted, kidney exchange programs are currently legal in many countries. These programs consist of a pool of incompatible patient-donor pairs and a set of compatible exchange between these pairs. The problem of maximizing the number of transplants in a kidney exchange program can be modeled as an integer programming problem (*e.g.*, see Abraham *et al.* [1]).

If the size of a pool of incompatible patient-donor pairs increases, it is expected that more transplantations can take place. Thus, it is relevant to study exchange programs involving several hospitals and countries. Note, however, that each hospital/country is a self-interested entity that aims to maximize her number of patients receiving a kidney. Thus, these situations can be studied from the game theory point of view. In [4, 5] an IPG modeling a kidney exchange game is studied.

5 Conclusions and further directions

Literature in non-cooperative game theory lacks the study of games with diverse sets of strategies with practical interest. This paper is a first attempt to address the computational complexity and existence of equilibria to integer programming games.

We classified the game's complexity in terms of existence of pure and mixed equilibria. For both cases, it was proved that the problems are Σ_2^p-complete. However, if the players' set of strategies is bounded, the game is guaranteed to have an equilibrium. Chen *et al.* [6] proved that computing an NE for a finite game is PPAD-complete even with only two players. Thus, recalling Figure 1, computing an NE to a separable IPG is PPAD-hard. Even when there are equilibria, the computation of one is a PPAD-hard problem, which is likely to be a class of hard problems. Furthermore, the PPAD class does not seem to provide a tight classification of the computational complexity of computing an equilibrium in IPGs. In fact, the PPAD class has its root in finite games that are an easier class of games, in comparison with general IPGs. Note that for IPGs, verifying if a profile of strategies is an equilibrium implies solving each player's best response optimization, which can be an NP-complete problem, while for finite games this computation can be done efficiently. In this context, it would be interesting to explore the definition of a "second level PPAD" class, that is, a class of problems for which a solution could be verified in polynomial time if there was access to an NP oracle.

In this paper, we also determined sufficient conditions for the existence of equilibria on IPGs. Moreover, these theoretical results enabled us to conclude that the support of an NE is finite. This is a key result in the correctness of the algorithm that computes an equilibrium for an IPG presented in [4]. Future work in this context should address the question of determining all equilibria, computing an equilibrium satisfying a specific property (*e.g.*, computing the equilibrium that maximizes the social welfare, computing a non-dominated equilibrium) and equilibria refinements or new solution concepts under a games with multiple equilibria. From a mathematical point of view, the first two questions embody a big challenge, since there seems to be hard to extract problem structure to the general IPG class of games. The last question raises another one, which is the possibility of considering different solution concepts to IPGs.

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