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DATA SCIENCE FOR REAL-TIME DECISION-MAKING

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July 2017

DS4DM-2017-008

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Abstract

A game merging the lot-sizing problem with a Cournot competition model is for the first time theoretically studied. Each player is a producer with her own production facility, modeled as an uncapacitated lot-sizing problem (*i.e.*, production incurs set-up and variable costs and inventories are allowed). A Cournot competition is played in each time period (market) with each player deciding the quantity of product to place on it. The market price of that product in each time period depends on the total quantity placed in the market.

We show that this game is potential with possibly multiple pure Nash equilibria. If the game has a single period, we prove that an equilibrium can be found in polynomial time, but it is weakly NP-hard to find an optimal pure Nash equilibrium (with respect to a given equilibrium refinement). If the game has no constant production and no inventory costs, we prove that a pure Nash equilibrium can be computed in polynomial time.

Keywords: Cournot competition; Lot-sizing problem; Nash equilibria; Potential game.

1 Introduction

The lot-sizing problem. Production planning is a classical studied problem in operations research, given its practical applications and the related challenging models in mixed integer optimization (see, *e.g.* [17]). The simplest case considers a firm with only one machine, planning the production of a single item. The lot-sizing problem (LSP) can be described as follows. There is a finite planning horizon. For each period there is a demand, a unit production cost (also known as variable cost) and a fixed set-up cost if production occurs. The goal is to find a production plan such that the demand of each period is satisfied and the total cost is minimized.

The Cournot competition. Cournot [3] developed one of the earliest examples of game analysis, now called the *Cournot duopoly*. In this setting, the players are two firms that produce a homogeneous good. They simultaneously choose their respective quantities to place in the market. Once the quantities in the market are known, the associated market price is determined. In this way, the profit of each firm depends on the strategy of the opponents. A profile of strategies for which both players simultaneously maximize their profits is called a *Cournot equilibrium* (later, this concept was generalized for any game and called *Nash equilibrium* [15]). The model can be extended to more than two firms, in a setting generally called *Cournot competition*

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(or *Cournot oligopoly*). In practice, the Cournot competition is a model mainly used to measure the market power of the participant firms.

Our game model. We build on the Cournot competition model of Pedroso and Smeers [16], where several producers compete for a single-product market over time. Our goal is to understand what is the likely behaviour of these producers (how much and when they will produce, how much and when they will place in the market) and what will be the resulting market prices. The natural conceptual framework for such a study is game theory under the key concept of Nash equilibrium.

Our model is a single-product market over a discrete and finite time horizon. At the beginning of the game, players simultaneously decide the quantities to introduce in the market of each period, taking into account both the Cournot competition taking place on each period and the production structure of each producer which is represented by an uncapacitated lot-sizing model.

Contributions and organization of the paper. We start with a literature review in Section 2. Section 3 provides some essential game theory background and formalizes the competitive uncapacitated lot-sizing game (ULSG) which is studied throughout the paper. Section 4 determines the best response functions of this game and a dynamic programming method to evaluate them in polynomial time. In Section 5 a result of Ui [22] is used to imply ULSG must have at least one pure Nash equilibrium (pNE), because it is a potential game. Section 6 and Section 7 describe polynomial time algorithms to find one such pNE in the single-period case and the no variable cost case, respectively. In addition, in Section 6 we show that finding the best pNE with respect to a given equilibrium refinement is weakly NP-hard, but one can compute such an optimal equilibrium in pseudo-polynomial time. In Section 8 extensions to our approach are considered: Section 8.1 remarks that our results can be easily extended if inventory costs exist, and Section 8.2 distinguishes other equilibria concepts. Section 9 summarizes the conclusions and open questions.

2 Literature Review

Most of the literature about lot-sizing games focuses on cooperative scenarios where, instead of searching for a Nash equilibrium, the goal is to find coalitions between the players such that they do not have incentive to leave them (as it would mean a utility decrease). See for example Wilco *et al.* [9]. Thain and Vetta [21] analyze predatory strategies, *i.e.*, where a firm with sufficient power plays to make the opponents' profit negative.

To the best of our knowledge, the literature on non-cooperative (competitive) lot-sizing games significantly differs from our setting. Typically, Cournot competitions models assume that the players profit functions are concave and that their restrictions (*e.g.*, production capacity) are linear. Therefore, Karush-Kuhn-Tucker (KKT) conditions (see Karush [10] and Kuhn and Tucker [11]) are enough to establish each player's optimization as a single system of inequalities and thus, computing a Nash equilibrium is simplified.

Maskin and Tirole [13] analyze an oligopoly, where set-up costs are considered and firms are committed to a particular action in the short-run. In opposition to the model that we present in this section, in [13], firms take decisions sequentially (period-by-period) and set-up costs are considered to be sufficiently large so that no two firms can operate profitably. Federgruen and Meissner [4] analyze a Bertrand (price) competition. In their model, each player decides a market price which is maintained throughout the game. Given these market prices, the demands in each time period for each player are determined. The authors are able to get sufficient conditions for the existence of an unique Nash equilibrium and an algorithm that efficiently computes a player best reaction if the set-up costs are constant during the whole time horizon for each player. It is also mentioned the Cournot competition associated with this model, where a player's strategy reduces to deciding an initial constant quantity which completely determines this player's demand market in each period of the market (basically, these demands vary by a multiplicative factor on that initial decision). Li and Meissner [12] consider a capacitated lot-sizing game in which the only strategic decision of each player is a production capacity which is to remain constant throughout the game. The capacity is purchased on a spot market before production decisions are made; its price depends on the total capacity acquired by the players. The authors prove the existence of a capacity equilibrium under modest assumptions. In the models of these two papers ([4] and [12]) the producers take their strategic decisions upfront (market price and production capacities, respectively) and commit to them during the entire planning horizon; we also focus on this static setting for determining equilibria on our model.

Recently, Gabriel *et al.* [6] built a methodology to solve Cournot competitions models when part of the players' decision variables are discrete, making the KKT-conditions unusable (which will be the case of our game). The authors propose a new approach that provides a compromise between complementarity and integrality, showing that a Nash equilibrium satisfying the integrality requirement can be found for a specific example. However, there is neither theoretical nor computational evidence showing the applicability of these ideas to the general case.

Pedroso and Smeers [16] apply a *tâtonnement process* in order to compute an equilibrium to the competitive lot-sizing game, *i.e.*, from an initial strategy for each player, iteratively, players deviate to a new strategy that improves their individual utility. In their computational experiments, this process successfully computes an equilibrium, opening the questions of what are sufficient conditions of convergence towards an equilibrium, how efficient this convergence would be and how it would vary according to the initialization strategies.

3 Preliminaries

The game model. We start by establishing the connection between the classical uncapacitated lot-sizing model (ULS) and the Cournot competition.

The model we build has a discretized finite time horizon with $T > 0$ periods. In each period t there is a market for a homogeneous product. We assume that for each period t, the market unit price is P_t, represented by the demand function $P_t = (a_t - b_t q_t)^+$ where $\alpha^+ = \max(\alpha, 0)$, q_t is the total quantity placed in the market, and a_t , b_t are given parameters modeling the market size and the level of players interaction, respectively. The set of firms (players) competing in this multi-period market is $M = \{1, 2, \ldots, m\}$. The production structure of each firm is represented by an uncapacitated lot-sizing model. That is, each firm p has to decide how much to produce in each time period t (production variable x_t^p) and how much to place in the market (variable q_t^p). For each firm p and period t, there are set-up and variable (linear) production costs, denoted by F_t^p and C_t^p, respectively, there is no upper limit on production quantities, and a producer can build inventory by producing in advance (the inventory variable for period t is h_t^p). We assume that there are no inventory costs (in Section 8.1 this assumption is removed). In this way, we obtain the following model for each player (firm) $p = 1, 2, \ldots, m$:

$$\max_{y^p, x^p, q^p, h^p} \quad \Pi^p(y^p, x^p, h^p, q^p, q^{-p}) = \sum_{t=1}^{T} P_t(q_t) q_t^p - \sum_{t=1}^{T} C_t^p x_t^p - \sum_{t=1}^{T} F_t^p y_t^p \tag{1a}$$

$$\text{s. t.} \quad x_t^p + h_{t-1}^p = h_t^p + q_t^p \qquad \text{for } t = 1, \ldots, T \tag{1b}$$

$$0 \leq x_t^p \leq B y_t^p \qquad \text{for } t = 1, \ldots, T \tag{1c}$$

$$h_0^p = h_T^p = 0 \tag{1d}$$

$$h_t^p, q_t^p \geq 0 \qquad \text{for } t = 1, \ldots, T \tag{1e}$$

$$y_t^p \in \{0, 1\} \qquad \text{for } t = 1, \ldots, T, \tag{1f}$$

where B is a sufficient large number such that Constraint (1c) is not bidding when $y_t^p = 1$, and $q_t = \sum_{i=1}^{m} q_t^i$ (total quantity introduced in the market of period t). The total quantity introduced in the market of period t is the responsible for the optimization program (1) to induce a game. The goal of player p is to maximize the utility (1a), which is simply the sum of her profit minus the production costs in each period t. Constraints (1b) represent the conservation of product. Constraints (1c) ensure that the quantities produced are non-negative and that whenever there is production ($x_t^p > 0$), the binary variable y_t^p is set to 1, implying the payment of the set-up cost F_t^p. We assume that the initial and final inventory quantities are zero, which is captured by equations (1d). Inventory quantities and output quantities must be non-negative, constraints (1e). The variables y_t^p are restricted to be binary through constraint (1f). We denote X^p as the set of feasible solutions for player p and by $X = \prod_{p=1}^{m} X^p$ the set of all feasible players' strategies.

Let y^p, x^p, h^p be T dimensional vectors of player p's decision variables for each time period t. Finally, for the sake of simplicity, let us assume that variable and set-up costs are positive integers, define producing in period $T+1$ as not participating in the game and/or $F^p_{T+1} = c^p_{T+1} = 0$.

Game theory background. Let the operator $(\cdot)^{-p}$ for some $p \in M$ denote $(\cdot)$ for all players except player p. A player p *best reaction* (or best response) to a (fixed) strategy q^{-p} of the opponents is a solution to Problem (1).

The concept of Nash equilibrium is widely accepted as solution for a game since it defines the most rational moves that the players can follow under a non-cooperative game. A *pure Nash equilibrium* for ULSG is a profile of strategies $(\overline{y}, \overline{x}, \overline{h}, \overline{q}) \in \prod_{p=1}^{m} X^p$ such that no player has incentive to deviate, this is, for each player p

$$\Pi^p(\overline{y}^p, \overline{x}^p, \overline{h}^p, \overline{q}^p, \overline{q}^{-p}) \geq \Pi^p(y^p, x^p, h^p, q^p, \overline{q}^{-p}) \quad \forall (y^p, x^p, h^p, q^p) \in X^p. \tag{2}$$

In other words, a pure Nash equilibrium (pNE in short) is an assignment to the variables (y, x, h, q) which is optimal for all the parametric programs 1. In a mixed Nash equilibrium, players choose a probability distribution over their set of strategies, which is computationally unsuitable when the description involves an exponential or infinite number of players' strategies. Moreover, as pointed out in Simon [20] and Rubinstein [19], players tend to prefer simple strategies. This reason together with the fact that ULSG has always a pure equilibrium (as we prove in Section 5), motivate us to concentrate our investigation only in pure Nash equilibria, assuming that players stay committed to the equilibrium strategy; in practice, however, each player may change her strategy in each period. In the beginning of Section 5, we will discuss the advantages and disadvantages of making such assumption.

The concept of potential game, as defined in Monderer and Shapley [14], is very useful when considering pure equilibria. The ULSG is a *potential game* if there is real-valued function Φ over the set of players feasible strategies $\prod_{p=1}^{m} X^p$ such that its value increases strictly when a player switches to a strategy that strictly increases her profit. This function is called potential. We will use to this property in Section 5.

4 Basic Properties

In the ULS problem the demand is fixed and the problem is reduced to minimizing the costs. A well-known and fundamental property of ULS is that it has an optimal solution with no stock at the begin of a period with positive production (see Pochet and Wolsey [17]). The same property holds for a player p's optimal solution to Problem (1): once q^p_t is determined, her problem reduces to solving an ULS.

Proposition 1 *Let $q^{-p} \in X^{-p}$ be fixed. There exists an optimal solution to* (1) *(a best response to q^{-p}) in which $h^p_{t-1} x^p_t = 0$ for $t = 1, 2, \ldots, T$.*

Proposition 1 is the essential ingredient to determine the optimal output quantities for player p.

Proposition 2 *Let $q^{-p} \in X^{-p}$ and player p's positive production periods $t_1 < t_2 < \ldots < t_r$ be fixed. There is an optimal solution to problem* (1) *satisfying*

$$\begin{aligned} &\overline{q}^p_t(q^{-p}) = 0, && \textit{for} \quad t = 1, 2, \ldots, t_1 - 1 \\ &\overline{q}^p_t(q^{-p}) = \frac{(a_t - b_t \sum_{i \neq p} q^i_t - C^p_{t_j})^+}{2b_t}, && \textit{for} \quad t = t_1, \ldots, T \\ \textit{with} \quad & j = max\{u : 1 \leq u \leq r, t_u \leq t\}. \end{aligned}$$

Proof. Let $T^p = \{t_1, t_2, \ldots, t_r\}$ be as stated in the proposition. By Proposition 1, in period $t \geq t_1$, the optimal output quantity q^p_t is produced in the latest production period t_j prior to t with positive production. Thus, the production variable can be simply replaced by $x^p_{t_j} = \sum_{t=t_j}^{\min(t_{j+1}-1, T)} q^p_t$. The optimal value for

q_t^p in (1) can be determined by optimizing an univariate concave quadratic function (the part of the utility function associated with q_t^p), that is,

$$(a_t - b_t q_t^p - b_t \sum_{i \neq p} q_t^i) q_t^p - C_{t_j}^p q_t^p$$

leading to the formulas of this proposition. □

The ULS can be solved in polynomial time through dynamic programming. If $q^{-p} \in X^{-p}$ is fixed, a similar idea extends to efficiently compute an optimal production plan for player p.

Lemma 1 *Solving player p's best reaction (1) for q^{-p} can be done in polynomial time.*

Proof. Let $G^p(t, q^{-p})$ be the maximum utility of player p over the first t periods, given the opponents' strategies q^{-p}. Then, $G^p(t, q^{-p})$ can be written as player p's maximum utility when the last production period was k

$$G^p(t, q^{-p}) = \max_{k: k \leq t} \{G^p(k-1, q^{-p}) + \sum_{u=k}^{t} (a_u - b_u(\overline{q}_u^p + \sum_{j \neq p}^{m} q_u^j))\overline{q}_u^p - F_k^p - C_k^p \sum_{u=k}^{t} \overline{q}_u^p\},$$

where $\overline{q}_u^p$ is computed according with Proposition 2. Thus, computing $G^p(T, q^{-p})$, which is equivalent to solve the best reaction problem (1) for q^{-p}, can be done in $O(T^2)$ time. □

In an equilibrium each player is selecting her best reaction (optimal solution of problem (1)) to the opponents' strategies on that equilibrium. Thus, once the players' production periods are fixed, we can apply Proposition 2 simultaneously for all the players, obtaining a system of equations in the output variables q which can be simplified and solved, resulting in the following proposition.

Proposition 3 *Let T^p be the set of periods with positive production for each player p for an ULSG. Then, an optimal output quantity for player p is*[1]

$$\overline{q}_t^p = 0, \qquad \textit{for} \quad t = 1, 2, \ldots, \min\{T^p\} - 1$$
$$\overline{q}_t^p = \frac{(P_t(S_t) - C_{\bar{u}^p}^p)^+}{b_t}, \qquad \textit{for} \quad t = \min\{T^p\}, \ldots, T,$$

where $\bar{u}^p = max\{u : u \in T^p, u \leq t\}$ (last production period prior to t for player p), $S_t = \{i : t \in T^i$ for $i = 1, 2, \ldots, m\}$ (players with positive production in period t) and $P_t(S_t) = \frac{a_t + \sum_{i \in S_t} C_{\bar{u}^i}^i}{|S_t|+1}$ (market price of period t). In particular, player p's utility is

$$\Pi^p(T^1, \ldots, T^m) = \sum_{t \in T^p} -F_t^p + \sum_{t=\min\{T^p\}}^{T} \frac{(P_t(S_t) - C_{\bar{u}^p}^p)^+}{b_t}(P_t(S_t) - C_{\bar{u}^p}^p). \tag{5}$$

In conclusion, the sets of periods with positive production for all the players are sufficient to describe a pNE. This fact significantly simplifies the game analysis in Section 6 and Section 7. In what follows, we use the notation of Proposition 3: S_t is the set of players participating in the market of period t and $P_t(S_t)$ is the unit market price of period t for the set of players S_t.

Proposition 3 leads to a natural variant of ULSG: restrict each player p's strategy to her set $T^p \subseteq \{1, \ldots, T, T+1\}$ of production periods and compute her utility according to utility (5); call this modified game by ULSG-sim. Proposition 3 associates output quantities to each profile of strategies in ULSG-sim. Because these output quantities are optimal for the fixed sets of production in ULSG-sim, the set of pNE of ULSG-sim propagates to the original ULSG:

[1]By optimal output quantities it must be understood the quantities of an pNE for the game in which production periods are fixed beforehand.

Proposition 4 *Any pNE of an ULSG-sim is a pNE of the associated ULSG.*

Therefore, ULSG can have a larger set of pNEthan ULSG-sim. In fact, there are pNE for ULSG that are not pNE of ULSG-sim. Example 1 shows this situation.

Example 1 *Consider the following instance with $m = 2$, $T = 2$, $a_1 = 12$, $a_2 = 9$, $b_1 = b_2 = 1$, $F_1^1 = 15, F_2^1 = 5$, $F_1^2 = 7$, $F_2^2 = 19$ and $C_1^1 = C_2^1 = C_1^2 = C_2^2 = 0$. Note that the absence of variable costs implies that it is a dominant strategy to produce only once.*

In the original game, $x^1 = q^1 = (0, \frac{a_2}{3b_2}) = (0, 3)$ and $x^2 = (\frac{a_1}{2b_1} + \frac{a_2}{3b_2}, 0)$, $q^2 = (\frac{a_1}{2b_1}, \frac{a_2}{3b_2}) = (6, 3)$ represents a profile of strategies that is a Nash equilibrium of ULSG with player 1's utility equal to 4 and player 2's utility equal to 38; if player 1 (player 2) does not participate in the game her utility decreases to zero, thus player 1 (player 2) does not have incentive to unilaterally deviate from the equilibrium and not produce; if player 1 decides to produce in period 1, then, by Proposition 2, she would produce $x^1 = (\frac{a_1}{4b_1} + \frac{a_2}{3b_2}, 0)$ and introduce in the market $q^1 = (\frac{a_1}{4b_1}, \frac{a_2}{3b_2})$, decreasing her utility to 3; if player 2 decides to produce in period 2, then, by Proposition 2, she would produce $x^2 = (0, \frac{a_2}{3b_2})$ and place on the market $q^2 = (0, \frac{a_2}{3b_2})$, decreasing her utility to -10.

Let us verify if the profile of strategies in ULSG-sim associated with the pNE to ULSG described above, $T^1 = \{2\}$ and $T^2 = \{1\}$, is a pNE for ULSG-sim. Player 1's utility for the profile of strategies under consideration is 4. Since player 1's utility is positive, the player has incentive to participate in the game. It remains to check if player 1 has incentive to produce in period 1. If player 1 deviates to $T^1 = \{1\}$ then the associated utility is $-F_1^1 + \frac{a_1^2}{9b_1^2} + \frac{a_2^2}{9b_2^2} = -15 + 16 + 9 = 10$ which is greater than when player 1 produces in period 2. Thus, $T^1 = \{2\}$ and $T^2 = \{1\}$ is not an equilibrium of ULSG-sim.

In Section 7, we compute a pNE for a special case of the ULSG-sim (and hence for the ULSG), using a potential function argument. The ULSG-sim, however, is not always a potential game like the ULSG (as we will show in the next section); this is illustrated by Example 2.

Example 2 *Consider the instance of ULSG-sim with $m = 2$, $T = 2$, $a_1 = 20$, $a_2 = 40$, $b_1 = b_2 = 1$, $F_1^1 = 17, F_2^1 = 10$, $F_1^2 = 18$, $F_2^2 = 10$, $C_1^1 = 7$, $C_2^1 = 5$, $C_1^2 = 17$ and $C_2^2 = 1$. The following relations for the players' utilities imply that a potential function Φ must satisfy $\Phi(\{1\}, \{1\}) < \Phi(\{1\}, \{1\})$ which is impossible:*

$$\begin{aligned}
\Pi^1(\{1\}, \{1\}) = \frac{8305}{36} &< \Pi^1(\{2\}, \{1\}) = \frac{2119}{9} \\
\Pi^2(\{2\}, \{1\}) = -\frac{83}{36} &< \Pi^2(\{2\}, \{3\}) = 0 \\
\Pi^1(\{2\}, \{3\}) = \frac{1185}{4} &< \Pi^1(\{1\}, \{3\}) = \frac{595}{2} \\
\Pi^2(\{1\}, \{3\}) = 0 &< \Pi^2(\{1\}, \{1\}) = \frac{7}{9}.
\end{aligned}$$

The discussion above clarifies the advantages and disadvantages of investigating ULSG through ULSG-sim.

5 Existence and Computation of Equilibria

Nash [15] proved the existence of a Nash equilibrium (pure or not) for any game with a finite number of players and finitely many strategies. However, in the ULSG the players' strategies are not finite and, therefore, Nash's result does not hold. In the literature that extends the discrete sets of strategies to continuous ones, there are results about the Nash equilibria existence. However, these results typically assume either well-behaved conditions on the players' profit functions, which do not hold for our model, or conditions in the game that are hard to verify. See for example the famous existence result by Arrow and Debreu [2].

If there were no set-up costs and $T = 1$, we would be under the classical Cournot competition where, clearly, the players with smallest variable costs will be the ones sharing the market; this will be treated in detail in Section 6. If we relax T to be arbitrary but keep the restriction of only variable costs, the problem is equivalent to solving the Cournot competition for each period t separately and considering the player p's variable cost in period t equal to $\min_{u=1,\dots,t} C_u^p$, this is, each player participates in market t by producing in advance in the least expensive period. In summary:

Theorem 1 *When $F_t^p = 0$ for $p = 1, 2, \dots, m$ and $t = 1, 2, \dots, T$, then the set of pNE for ULSG, projected onto the variables (x, h, q) is contained in a polytope and the market price is equal for all the pNE. Furthermore, unless the problem is degenerate (*i.e.., *there are at least two players for which the production costs coincide with the market price in an equilibrium), there is only one pNE, and it can be computed in polynomial time.*

Next, we investigate the effect on the equilibria search when set-up costs are introduced in the game.

In what follows we show that our game possesses at least one pNE through the concept of potential game.

Proposition 5 *The ULSG is a potential game that contains pNE, one of them being a maximizer in X of the game potential function*

$$\Phi(y,x,h,q) = \sum_{p=1}^{m}\sum_{t=1}^{T}\left[-F_t^p y_t^p - C_t^p x_t^p + \left(a_t - \frac{b_t}{2}(2q_t^p + \sum_{i\neq p} q_t^i)\right) q_t^p\right] \tag{7a}$$

$$= \sum_{p=1}^{m}\left[\Pi^p\left(y^p, x^p, q^p, q^{-p}\right) + \sum_{t=1}^{T}\left(\frac{q_t^p b_t}{2}\sum_{i\neq p} q_t^i\right)\right]. \tag{7b}$$

Proof. The fact that ULSG is a potential game with the exact potential function (7) is a direct result from Ui [22]. Ui explicitly gives the potential function for any Cournot competition where the profit functions u on the strategies x have the form $u^p(x^p) = \sum_{j\neq p} w^{j,p}(x^j, x^p) - h^p(x^p)$, for functions $w^{j,p}, h^p$ satisfying the symmetry condition $w^{j,p}(x^j, x^p) = w^{p,j}(x^p, x^j)$ for all x^j, x^p, $j \neq p$.

It is also well known that a strategy maximizing the potential function of a potential game is a pure Nash equilibrium (Monderer and Shapley [14]). More generally, if we define the neighborhood of a point $(y, x, h, q) \in X$ to be any point in X such that only one player modifies her strategy then, any local maximum of the potential function $\Phi(y, x, h, q)$ is a pNE. It only remains to check that the potential function Φ has indeed a maximum in the domain of feasible strategies. This follows from the fact that Φ is a linear combination of binary variables (and hence, bounded) plus a concave function (see Appendix A). □

Given that ULSG is potential and its potential function has an optimum, a pNE can be found: assign a profile of strategies for the players; while there is a player with incentive to unilaterally deviate from the current profile of strategies, replace her strategy by one that improves that player's utility; when no player can improve, an equilibrium was found. This is called a tatônnement process or adjustment process, which for ULSG is guaranteed to converge to a pNE, since in each iteration the value of the potential function strictly increases. Although each iteration of the process can be performed in polynomial time (Lemma 1), we could not prove that the number of iterations is polynomial in the size of the input, which would imply that the tatônnement process runs in polynomial time.

Alternatively, in order to find an equilibrium, one could compute a maximum of the potential function $\Phi(y, x, h, q)$ in X which amounts to solve a concave mixed integer quadratic programming probme (MIQP), see the proof in Appendix A. Once the binary variables y are fixed, *i.e.*, production periods have been decided, maximizing the potential function amounts to solve a concave quadratic problem and therefore, a maximum can be computed efficiently. In particular, recall from Theorem 1, that if there are no set-up costs (which is equivalent to say that the binary variables y_t^p are set to zero and constraints (1c) are removed) there is (in general) a unique equilibrium which can be found in polynomial time. Once set-up costs are considered, the analyses seems to complicate as indicated by the fact that a player's advantage in the game is not anymore a mirror of her variable cost alone. Since computing an equilibrium through the potential function maximization implies solving an MIQP which in general is hard, we will restrict our study to simpler cases (single period and only set-up costs) in an attempt to get insight in the understanding of the game's equilibria.

6 Single Period Case

In all this section we restrict our attention exclusively to the case with a single period ($T = 1$). For simplicity, we drop the subscript t from our notation. Note that in this setting the quantities produced are exactly those placed in the market ($x = q$). Also, by Proposition 3, the problem of computing equilibria reduces to deciding the set of players producing strictly positive quantities. We show that characterizing the set of pNE is a weakly NP-complete problem (in a sense to be defined), that admits a pseudopolynomial time algorithm. Moreover, we can find one such pNE in polynomial time. All these results follow from a simpler characterization of the equilibrium conditions that we now describe.

In a pNE, a subset of producers $S \subseteq \{1, 2, \ldots, m\}$ play a strictly positive quantity. By the definition of pNE, no player in S has incentive to stop producing (leave S) and a player not in S has no incentive to start producing (enter in S). Therefore, applying Proposition 3, a player p in S must have non-negative utility

$$-F^p + \frac{(P(S) - C^p)^+}{b}(P(S) - C^p) \geq 0 \quad \Leftrightarrow \quad P(S) \geq \sqrt{F^p b} + C^p, \tag{8}$$

while a player p not in S must have non-positive utility if she enters S, even if producing the optimal quantity $\frac{(P(S)-C^p)^+}{2b}$ given by Proposition 2

$$-F^p + \frac{(P(S) - C^p)^+}{2b}\frac{(P(S) - C^p)}{2} \leq 0 \quad \Leftrightarrow \quad P(S) \leq 2\sqrt{F^p b} + C^p. \tag{9}$$

To find one pNE efficiently, we propose Algorithm 6.0.1. In a nutshell, this algorithm uses the lower bounds to $P(S)$ given by conditions 8 to order the players in step 1. Starting from $S = \emptyset$, it adds a player to S whenever she has advantage in joining the current S (step 4). Since a player p will only join S if her variable cost C^p is no larger than the market price, it is easy to see that $P(S)$ decreases whenever a player is added to S (note that according with Section 4 $P(S)$ is simply the average of the variable costs together with the parameter a). Thus, once in iteration k, if player p did not had incentive to enter S then, she will never have it in the future updates of S. On the other hand, taking into account the order of the players, whenever player p has incentive to be added to S, we have $P(S \cup \{p\}) > \sqrt{F^p b} + C^p \geq \sqrt{F^i b} + C^i$ for all $i \in S$, ensuring condition (8). This shows that the algorithm outputs correctly a pNE.

In Algorithm 6.0.1, step 1 involves ordering a set of m numbers, which can be done in $O(m \log m)$ time. Then, a cycle which can cost $O(m)$ time follows. In this way, it is easy to conclude that the algorithm runs in time $O(m \log m)$.

Theorem 2 *Algorithm 6.0.1 outputs a pNE and runs in $O(m \log m)$ time.*

In particular, the last theorem implies that there is always at least one pNE. To see that there can be more than one, consider an instance where all players have $C^p = 0$ and $F^p = F$. Then Algorithm 6.0.1 will stop adding elements when $P(S) = a/(|S| + 1) < 2\sqrt{Fb}$. But since the order is arbitrary, this means that any set S of cardinality $\lceil a/(2\sqrt{Fb})\rceil - 1$ is a pNE. Therefore, an alternative to overcome the multiplicity of equilibria would be to search for the best pNE according to a given criteria. In other words, the goal would be to determine the pNE that maximizes $\sum_{i \in S} v^i$, where S is the set of players producing in the pNE. The decision version of this problem is the following:

Problem: Optimize 1-Period Uncapacitated Lot Sizing Game

Algorithm 6.0.1

Require: A single period ULSG instance.
Ensure: A subset S of players producing strictly positive quantities in a pNE.
Assume that the players are ordered according with $\sqrt{F^1 b} + C^1 \leq \sqrt{F^2 b} + C^2 \leq \ldots \leq \sqrt{F^m b} + C^m$.
Initialize $S \leftarrow \emptyset$
for $1 \leq p \leq m$ **do**
  if $C^p + 2\sqrt{F^p b} < P(S)$ **then**
    $S \leftarrow S \cup \{p\}$
  else
    if $P(S \cup \{p\}) \geq \sqrt{F^p b} + C^p$ **then**
      Arbitrarily decide to set p in S.
    end if
  end if
end for
return S

Instance: Positive reals a, b, B, vectors $C, F \in \mathbb{Z}^m_+$ and $v \in \mathbb{Z}^m$. (1P-LSG-OPT)

Question: Is there a subset S of $\{1, 2, \ldots, m\}$ such that

$$\sum_{i \in S} v^i \geq B \tag{10a}$$

$$C^p + \sqrt{F^p b} \leq P(S) \quad \forall k \in S \tag{10b}$$

$$C^p + 2\sqrt{F^p b} \geq P(S) \quad \forall k \notin S \quad ? \tag{10c}$$

It turns out that 1P-LSG-OPT is NP-complete and thus, likely to be an intractable problem. We prove this through a reduction from PARTITION (given a set of n positive integers, find if they can be split into two groups with identical sum), which is weakly NP-complete [7].

Theorem 3 1P-LSG-OPT *is NP-complete.*

Proof. Given a set $S \subseteq \{1, 2, \ldots, m\}$, constraints (10a), (10b) and (10c) can be verified in polynomial time in the size of the instance. Therefore, 1P-LSG-OPT is in NP.

We show that 1P-LSG-OPT is NP-complete by a reduction from PARTITION. Let $\{a_i\}_{i=1..m}$ be an instance of PARTITION. Set $A = \frac{1}{2}\sum_{i=1}^m a_i$ and $M = 1 + 2A$. We construct the following instance of 1P-LSG-OPT.

- Set $b = 1$, $a = Am$, and $B = M - A$.
- $\mathcal{I} = \{1, 2, \ldots, m\}$ is a set of m players such that for each element $i = 1, 2, \ldots, m-1$ set $C^i = a_i$, $F^i = (A - C^i)^2$ and $v^i = -a_i$, and $C^m = a_m$, $F^m = (A - C^m)^2$ and $v^m = -a_m + M$.
- $\mathcal{D} = \{m+1, m+2, \ldots, 2m-1\}$ is a set of $m-1$ dummy players such that for each element $i = m+1, m+2, \ldots, 2m-1$ set $C^i = 0$, $F^i = \left(\frac{A}{2}\right)^2$ and $v^i = 0$.
- Set an *upper bound* player UB with $C^{\text{UB}} = A$, $F^{\text{UB}} = 0$ and $v^{\text{UB}} = -3M$.

(Proof of if). For a YES instance of PARTITION, there is $Z \subseteq \{1, 2, \ldots, m\}$ so that $\sum_{i \in Z} a_i = A$ and $m \in Z$. Note that $S = Z \cup \{m+1, m+2, \ldots, 2m - |Z|\}$ is a solution to 1P-LSG-OPT, with $|S| = m$, and whose market price $P(S)$ equals

$$\frac{a + \sum_{i \in S} C^i}{|S| + 1} = \frac{Am + \sum_{i \in Z} a_i}{m + 1} = \frac{Am + A}{m + 1} = A.$$

Let us verify that the S is indeed a YES instance for 1P-LSG-OPT.

Inequality (10a) is satisfied: since $m \in Z \subseteq S$, then

$$\sum_{i \in S} v^i = M - \sum_{i \in Z} a_i = M - A = B.$$

Inequalities (10b) hold for S:

$$C^p + \sqrt{F^p b} = a_p + \sqrt{(A - a_p)^2} = A = P(S), \ \ \forall p \in S \cap \mathcal{I} = Z$$
$$C^p + \sqrt{F^p b} = 0 + \sqrt{\left(\frac{A}{2}\right)^2} = \frac{A}{2} \leq P(S), \ \ \forall p \in S \cap \mathcal{D}.$$

Inequalities (10c) hold: using $a_p < A$ for $p = 1, 2, \dots, m$, it follows that

$$C^p + 2\sqrt{F^p b} = 2A - a_p \geq A = P(S), \ \ \forall p \in \mathcal{I} \setminus S$$
$$C^p + 2\sqrt{F^p b} = A = P(S), \ \ \forall p \in \mathcal{D} \setminus S$$
$$C^{\mathrm{UB}} + 2\sqrt{F^{\mathrm{UB}} b} = A = P(S).$$

(Proof of only if). It is easy to check that the v^i values and B are set in such a way that any YES instance S of 1P-LSG-OPT must contain player m, but cannot contain the upper bound player UB.

Using inequalities (10b) and (10c) for players m and UB, respectively, it follows that $P(S)$ must be equal to A. In particular

$$P(S) = A \Rightarrow \frac{Am + \sum_{i \in S \cap \mathcal{I}} a_i}{|S| + 1} = A \Rightarrow \sum_{i \in S \cap \mathcal{I}} a_i = A(|S| + 1 - m).$$

Since $m \in S$, $\sum_{i \in S \cap \mathcal{I}} a_i > 0$ and thus, the right-hand-side above is positive, leading to $|S| > m-1$. Moreover, since this is a YES instance $\sum_{i \in S \cap \mathcal{I}} a_i \leq A$ and thus $|S| \leq m$. Therefore, $|S| = m$ and $\sum_{i \in S \cap \mathcal{I}} a_i = A$. □

Theorem 3 shows that maximizing a linear function over the set of pNE is hard, assuming $P \neq NP$. Yet, we can build a pseudo-polynomial time algorithm to solve this problem if C^i are integer: let $L_p = C^p + \sqrt{F^p b}$ and $U_p = C^p + 2\sqrt{F^p b}$ for $p = 1, 2, \dots, m$. We propose to solve this problem using Algorithm 6.0.2, where $H(k, l, r, s, C)$ is the optimal value of the problem limited to players $\{1, 2, \dots, k\}$, where $|S| = l$, the tightest lower bound is L_r, the tightest upper bound is U_s and $C = \sum_{i \in S} C^i$.

From each (k, l, r, s, C), we can choose to either add $k + 1$ or not to the set S, leading to the updates of lines 3 and 4, respectively. At the end, the optimal objective function value is given by the maximum entry $H(m, l, r, s, C)$ leading to a feasible solution. It is easy to build the optimal S by a standard backward pass of the underlying recursion.

Algorithm 6.0.2

Require: A single period ULSG instance and a vector $v \in \mathbb{Z}^m$.
Ensure: The optimal value of the input function associated with p over the set of pNE.

1: Initialize $H(\cdot) \leftarrow -\infty$ but $H(0, 0, 0, 0, 0) \leftarrow 0$.
2: **for** $k = 0$ to $m - 1$; $l, r, s = 0$ to k; $C = 0$ to $\sum_{i=0}^{k} C^i$ **do**
3: $\quad H(k+1, l+1, \arg\max_{i=k+1,r} L_i, s, C + C^k) \leftarrow$
$\quad\quad \max(H(k+1, l+1, \arg\max_{i=k+1,r} L_i, s, C + C^k), H(k, l, r, s, C) + v^{k+1})$
4: $\quad H(k+1, l, r, \arg\min_{i=k+1,s} U_i, C) \leftarrow$
$\quad\quad \max(H(k+1, l, r, \arg\min_{i=k+1,s} U_i, C), H(k, l, r, s, C))$
5: **end for**
6: **return** $\arg\max_{l,r,s,C}\{H(m, l, r, s, C) | L_r \leq \frac{a+C}{l+1} \leq U_s\}$.

Therefore we have established the following result.

Theorem 4 *If C^i integer for $i = 1, \dots, m$, finding the optimal pNE in the 1-period lot-sizing game can be solved in $\mathcal{O}(m^4 \sum_{k=1}^m C^k)$ time.*

The potential function (7) restricted to this case, *i.e.*, $T = 1$ and domain 2^m (power set of $\{1, 2, \dots, m\}$), is submodular. Thus, an alternative could be to solve to maximize this submodular function. It is well-known that submodular functions are hard to maximize. This is the reason why we built an algorithm to compute a pNE which is not based on this function.

7 Congestion Game Equivalence

Throughout this section, we approach the ULSG with only set-up costs, *i.e.*, $C_t^k = 0$ for all $k = 1, 2, \dots, m$ and $t = 1, 2, \dots, T$. There are two immediate important observations valid in this special case. One is that it is always optimal for a player to produce only once in order to minimize the set-up costs. Another is that the strategies in a pNE depend only on the number of players sharing the market in each period. From Proposition 3, if S_t are the players participating in period t, then their revenue is $\frac{a_t^2}{b_t(|S_t|+1)^2}$, with a market price of $P_t(S_t) = \frac{a_t}{|S_t|+1}$.

These observations lead to a connection with congestion games. A *congestion game* is one where a collection of players has to go from a (source) vertex in a digraph to another (sink) and the cost of using an arc of the graph depends on the number of players also selecting it in their paths; each player's goal is to minimize the cost of her path; see Rosenthal [18]. We can easily reformulate ULSG-sim as a congestion game: consider a digraph $G = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N} = \mathcal{S} \cup \mathcal{T}$ with $\mathcal{S} = \{s_1, s_2, \dots, s_m\}$ and $\mathcal{T} = \{1, 2, \dots, T, T+1\}$, and $\mathcal{A} = \mathcal{F} \cup \mathcal{P}$ with $\mathcal{F} = \{(s_k, t) : k = 1, 2, \dots, m \text{ and } t = 1, 2, \dots, T+1\}$ and $\mathcal{P} = \{(t, t+1) : t = 1, 2, \dots, T\}$. The cost of arcs $(s_k, t) \in \mathcal{F}$ equals F_t^k; the cost of arcs $(t, t+1) \in \mathcal{P}$ equals $-\frac{a_t^2}{b_t(1+n)^2}$, where n is the number of players selecting this arc. Finally, for each player k the source vertex is s_k and the sink is $T + 1$. Figure 1 illustrates this transformation. This reformulation has polynomial size since, the number of vertices is $m + T + 1$ and the number of arcs is $m(T + 1) + T$ (note that the size of ULSG is $O(mT)$ since mT set-up costs are given).

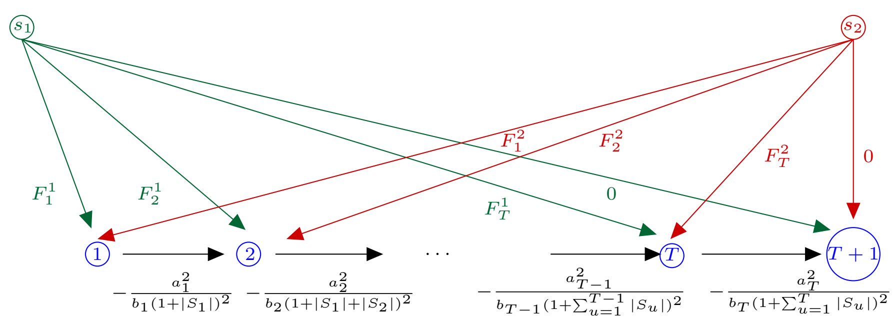

Figure 1: Congestion game for ULSG-sim with $m = 2$.

Any congestion game is a potential game, as proved by Rosenthal [18] (the converse is also true, see Monderer and Shapley [14]). Rosenthal [18] provides a potential function which in our case is

$$\Phi(t_1, \dots, t_m) = \sum_{k=1}^{m} -F_{t^k}^k + \sum_{t=1}^{T} \sum_{k=1}^{n_t} \frac{a_t^2}{(k+1)^2 b_t}, \tag{14}$$

where $t_k \in \{1, 2, \dots, T+1\}$ is the period in which player k produces and n_t is the number of players producing before or at period t (*i.e.*, the cardinality of the set $\{k : t_k \le t, k = 1, \dots, m\}$). Using the same argument as in proof of Proposition 7, one can prove that a maximizer of 14 is a pNE for ULSG-sim and thus, by Proposition 4, for ULSG.

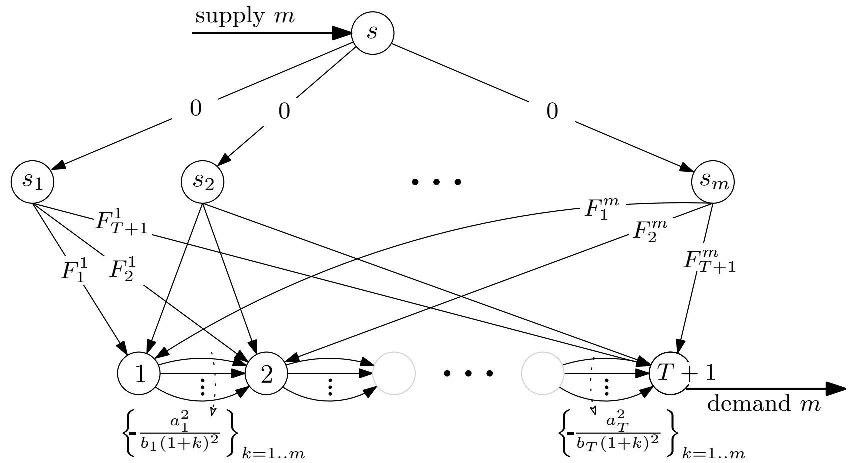

Figure 2: Minimum cost flow approach to optimize (14). All arcs have unit capacity.

For this specific problem, maximizing the potential function (14) is equivalent to solving the minimum cost flow problem in the following network (see Figure 2):

- consider a digraph $G = (\mathcal{N}', \mathcal{A}')$ where $\mathcal{N}' = \{s\} \cup \mathcal{S} \cup \mathcal{T}$ with $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$ and $\mathcal{T} = \{1, 2, \ldots, T, T+1\}$, and $\mathcal{A}' = \mathcal{I} \cup \mathcal{F} \cup \mathcal{P}'$ with $\mathcal{I} = \{(s, s_k) : k = 1, 2, \ldots, m\}$, $\mathcal{F} = \{(s_k, t) : k = 1, 2, \ldots, m \text{ and } t = 1, 2, \ldots, T+1\}$ and $\mathcal{P}' = \{(t, t+1) : t = 1, 2, \ldots, T \text{ and } k = 1, \ldots, m\}$ (m parallel arcs).
- for $(s, s_k) \in \mathcal{I}$ the cost is 0 and capacity is 1;
- for $(s_k, t) \in \mathcal{F}$ the cost is F_t^k and capacity is 1; set $F_{T+1}^k = 0$;
- for $(t, t+1) \in \mathcal{P}'$ and $k = 1, \ldots, m$, the cost is $-\frac{a_t^2}{b_t(1+k)^2}$ and capacity is 1;
- the supply is m in vertex s and the demand at $T+1$ is m.

Observe that this reformulation is polynomial in the size of an ULSG instance: the network has $1+m+T+1$ vertices and $m + m(T+1) + mT$ arcs. The advantage of this reformulation is that solving a min-cost flow problem can be done in polynomial time (see Goldberg and Tarjan [8]). The solution of this minimum cost flow problem is the optimal flow in each arc of the graph 2. From this solution, we extract the players' production periods in an equilibrium of ULSG-sim: the arcs in $\mathcal{F}$ with positive flow (in particular, the flow is 1 since the solution is integer and respects the arcs capacity) give the periods for each player to produce.

There is an alternative approach to compute a, possibly distinct, pNE. A maximum of the potential function (7) is a pNE and it is in the subset of strategies in which the players decide the production period and choose the optimal quantities according to Proposition 3. Therefore, restricting function (7) to this subset of strategies, it simplifies to

$$\Phi(t_1, t_2, \ldots, t_m) = \sum_{p=1}^{m} \left(-F_{t_p}^p + \sum_{t=t_p}^{T} \frac{a_t^2}{b_t(n_t \ +1)^2} + \sum_{t=t_p}^{T} \frac{a_t^2}{2(n_t+1)} (n_t - 1) \frac{a_t^2}{(n_t+1)b_t} \right)$$

$$= \sum_{p=1}^{m} -F_{t_p}^p + \sum_{t=1}^{T} \frac{a_t^2}{2b_t(n_t+1)} n_t \tag{15a}$$

$$= \sum_{p=1}^{m} -F_{t_p}^p + \sum_{t=1}^{T} \sum_{i=1}^{n_t} \frac{a_t^2}{2i(i+1)b_t}. \tag{15b}$$

Once again, computing the maximum of (15b) is equivalent to solving a minimum cost flow problem similar to the one in Figure 2 (the difference is in the cost of the arcs $(t, t+1)$ which are $\{\frac{a_t^2}{2k(k+1)b_t}\}_{k=1,\ldots,m}$ for $t = 1, \ldots, T$).

We remark that there are instances for which the optimal solutions for the maximums of 14 and 15b do not coincide and thus, two possibly distinct pNE can be computed in polynomial time.

The results of this section are summarized in the following theorem.

Theorem 5 *When $C_t^k = 0$ for $k = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, T$, a pNE for an ULSG can be computed in polynomial time by solving a minimum-cost network flow problem.*

8 Extensions

8.1 Inventory costs

In the lot-sizing problem, inventory costs must in general, be taken into account. It is a natural aspect in real-world applications that influences the optimal production plans. However, if inventory costs for each period are considered in the objective function, using the flow conservation constraints, an uncapacitated LSP can be transformed in an equivalent one without inventory costs, which are included in the updated variable production costs. In ULSG, if each player p's objective (1a) considers inventory costs H_t^p for each period t, an analogous replacement of the inventory variables h_t^p (through constraint (1b)) results in updated variable production costs, but also in new market prices; these market prices now depend on each player's inventory costs; therefore, since in the results previously presented we consider equal market prices for each player, the inclusion of inventory costs requires their adaption.

Proposition 6 *Consider an ULSG with each player p's utility function equal to*

$$\Pi^p(y^p, x^p, h^p, q^p, q^{-p}) = \sum_{t=1}^{T} P_t(q_t)q_t^p - \sum_{t=1}^{T} C_t^p x_t^p - \sum_{t=1}^{T-1} H_t^p h_t^p - \sum_{t=1}^{T} F_t^p y_t^p. \tag{16}$$

The results presented in Section 4 and Section 5 for each player p hold if the market size parameter a_t is replaced by $a_t^p = a_t + \sum_{u=t}^{T-1} H_u^p$, variable cost C_t^p is replaced by $\hat{C}_t^p = C_t^p + \sum_{u=t}^{T-1} H_u^p$ and the market price $P_t(S_t)$ is replaced by $P_t^p(S_t) = a_t^p + \frac{\sum_{i \in S_t} \left(\hat{C}_{t_j^i}^i - a_t^i \right)}{|S_t|+1}$.

Proof. One can use constraints (1b) to eliminate the inventory variables in player p's objective function (16). Thus, using $h_t^p = \sum_{u=1}^{t} (x_t^p - q_t^p)$ in the objective function (16), leads to

$$\Pi^p(y^p, x^p, h^p, q^p, q^{-p}) = \sum_{t=1}^{T} (a_t^p - b_t q_t)^+ q_t^p - \sum_{t=1}^{T} \hat{C}_t^p x_t^p - \sum_{t=1}^{T} F_t^p y_t^p.$$

and the proof follows. □

8.2 Other equilibria

According with Section 3, the players stay committed to a strategy from the beginning of the game, the firms play simultaneous their quantities in each period and their objective is to maximize their own profit. However, these assumptions might not hold: (i) firms can change their strategies in each time period or (ii) there can be a firm (leader) that selects her quantity in the market before the other participants (followers). Thus, other equilibria can arise in the game. We will illustrate these alternative equilibria through examples.

8.2.1 Subgame-perfect equilibrium

Strategies can incorporate dynamic decisions which are dependent of what has been observed in past periods. This dynamic allows firms to reach equilibrium strategies that can be substantially different with respect to those observed on static games.

We will restrict to symmetric lot-sizing games with 2 players and 2 periods. If a, b, c and F defines one of these periods, a static equilibrium could arise when one of the players plays:

- the monopoly quantity to place in the market is $\frac{a-c}{2b}$ with profit $\frac{(a-c)^2}{4b} - F$
- the duopoly quantity to place in the market is $\frac{a-c}{3b}$ with profit $\frac{(a-c)^2}{9b} - F$

On the other hand, if the firms cooperate in order to maximize their total profit in each period, each of them plays $\frac{(a-c)}{4b}$ for a profit $\frac{(a-c)^2}{8b} - F$. Achieving cooperation even at one period is not possible on static games.

With dynamic strategies, the firms can achieve cooperation. Let us consider an instance of ULSG with market $a_2 = a_1 = a_2 M = 24M$, $b_1 = b_2 M^2 = M^2$ and production costs $F_1^1 = F_2^1 = 0$, $F_1^2 = F_2^2 = 49$, $C_1^1 = C_2^1 = 20M$ and $C_1^2 = C_2^2 = 0$. The constant $M = 10$ is not substituted for convenience. The game proceeds in two rounds. First, firms decide simultaneously their production and market placing in period 1. After these decisions are revealed, firms decide simultaneously their production and market placing in period 2.

In period 2 the monopoly and duopoly quantities are 12 and 8 respectively, leading to profits of $144-49 > 0$ and $64 - 49 > 0$. It can be easily checked that both cases lead to a Nash Equilibrium. Consider now the following "dynamic" strategy:

- Play the cooperative quantity in period $t = 1$. This means, to produce and place in the market $q = \frac{(a_1-c_1)}{4b_1} = 1/M$.
- If in period $t = 1$ both players cooperated, play the duopoly quantity ($q = 8$) in period two. Otherwise,
  - If you cooperated, play to reach the Nash equilibrium of the subgame in the second period that minimizes the profit of the other player.
  - If the other player cooperated, play to reach the Nash equilibrium of the subgame in the second period that minimizes your own profit.
  - If neither player cooperated, play to reach any predefined Nash equilibrium, e.g., the one that minimizes the difference of profits of the combined game.

If both players follow this strategy, they will cooperate in period 1 and then play the duopoly quantity in period 2. Note that this is strictly better than playing the duopoly quantities in both periods (which is the closest equilibrium you can obtain in the static case). The strategy relies on multiple equilibria to establish a threat: a firm offers to cooperate in period 1; however, if the other does not cooperate, the firm will select in period 2 the Nash equilibrium that will minimize the profit of the deviating firm.

We will show that this strategy leads to a subgame perfect equilibrium [5]. This is a refinement of the Nash equilibrium concept that enforces the credibility of the threats that are or could be played at every period. It implies that no matter what players do in period 1, they must play to reach a Nash Equilibrium in period 2. We did not explicitly specify the equilibria to be played in period 2 because Proposition 2 does not necessarily apply to dynamic games, since a player could have incentive to hold inventory from period 1 to period 2 for strategic reasons (*e.g.*, to prevent an equilibrium from happening).

We will prove that this strategy induces a subgame perfect equilibrium by using the one shot deviation principle [5], which basically allows us to look at deviations from the equilibrium strategy that only change in one period. Because the game is symmetric, we will assume that player 2 sticks to the strategy, while player 1 attempts to deviate in order to increase her profit:

- Deviations in period 2: if player 1 follows the strategy in period 1, then cooperation is achieved. In period 2, the player 1 has no incentive to deviate from their strategy, since they are playing the duopoly equilibrium.
- Deviations in period 1: Because player 1 is deviating in period 1, she may want to build some inventory $\bar{q}$ in period 1 to place in the market of period 2. We claim that in any profitable deviation, $\bar{q}$ is bounded by a constant. Indeed, each unit held costs $20M$ to produce, which is then sold by at most 24 per unit. So there is at least a loss of $20M - 24 = 176$ per unit held. Even as a monopoly, the most that player 1 can get in both periods is upper bounded by 4+144 =148, so holding $\bar{q}$ cannot be convenient for $\bar{q} \geq 148/174$.

Because player 1 follows the strategy in period 2, but does not cooperate in period 1, then in period 2 the Nash equilibrium that minimizes the profit of player 1 will be played. The worst possible Nash equilibrium for player 1 is one in which she does not produce: player 2 will try to maximize $(24-(\bar{q}+q))q$, so $q = 12 - \bar{q}/2$. Since player 1 places $\bar{q} < 148/174 < 2$, the profit of player 2 in period 2 is at least $(12-\bar{q}/2)^2 - 49 > 11^2 - 49 > 0$, so it is profitable for player 2 to enter the market in period 2. It is easy to check that the best response to this quantity is actually not produce (even with $\bar{q} = 0$ the profit is upper bounded by $36 - 49 < 0$), so indeed this leads to a Nash equilibrium in which player 1 does not produce in period 2.
Therefore, from the perspective of player 1, deviating in period 1:

– It can give an increased profit in period 1. This profit can increase from 2 (playing cooperatively) to 9/4 playing the best response to the cooperative quantity. So the increase is upper bounded by 1/4.

– Leads to player 1 not to produce in period 2. This causes a loss of the entire duopoly profit of $64 - 49 = 15$.

– Holding inventory only leads to further losses.

It follows that deviating in period 1 is not profitable.

8.2.2 Stackelberg equilibrium

Consider the instance of ULSG with $m = 3$, $T = 1$, $a_1 = a$, $b_1 = b$, $F_1^1 = F_1^2 = F_1^3 = 0$, $C_1^1 = C_1^2 = C_1^3 = c$. Let firm 1 be the leader (plays first) and firm 2 and 3 are the followers. Firms 2 and 3 play simultaneously, thus, according with 2, they produce $q^2 = \frac{a-b(q^1+q^3)-c}{2b}$ and $q^3 = \frac{a-b(q^1+q^2)-c}{2b}$, respectively. Therefore, $q^2(q^1) = \frac{a-bq^1-c}{3b}$ and $q^3(q^1) = \frac{a-bq^1-c}{3b}$. Since firm 1 plays first and knows the optimal strategies for firm 2 and 3 according with her quantity q^1, her optimal strategy is $q^1 = \frac{a-c}{2b}$ which leads firm 2 and 3 to play $\frac{a-c}{6b}$.

This example shows the advantage of the leader of influencing the equilibrium strategies of the followers.

9 Conclusions and open problems

In the uncapacitated lot-sizing game, the production cost of player p in period t depends on two parameters: the variable cost C_t^p and the set-up cost F_t^p. When we consider production costs with only one of these parameters or a single period, the problem of computing a pure equilibrium becomes tractable, although characterizing the set of pure equilibria is NP-complete. The question of whether finding an optimal pNE

Problem	Compute one pNE	Characterize the set of pNE
ULSG with $T = 1$	P	NP-complete
ULSG with $F = 0$	P	P
ULSG with $C = 0$	P	?
ULSG	?	NP-complete

Table 1: Computational complexity of ULSG.

is a tractable problem when there are no variable costs remains open. Another open question is whether a pNE can be efficiently computed for the general case (*i.e.*, more than one period and no restriction on the costs). Table 1 summarizes our findings.

A typical constraint in the lot-sizing problem is the presence of positive initial and final inventory quantities, which for the uncapacitated case can be assumed to be 0, without loss of generality, by modifying the demands (see Pochet and Wolsey [17]). In one hand, considering positive initial and final inventory quantities in ULSG for each player does not interfere with the fact that the game is potential, since the the objective function does not change. On the other hand, this is problematic when characterizing each player's best response, since in the game there is no fixed demand to satisfy. Therefore, it is interesting to study the influence of relaxing the assumption that initial and final inventories are zero in future research.

When production capacities are introduced in LSP, it becomes NP-complete (see [17]). Thus, if there are players' production capacities for each period in our game, solving each player's best response becomes NP-complete. Note that this does not interfere in the formulation of a player's utility function, and thus the game remains potential with only the potential function domain reduced by the limitations imposed buy the production capacity.

Therefore, including more restrictions (*e.g.* positive initial and/or final inventory quantities, production capacities) on the lot-sizing model of each player will not change the fact that the game is potential and thus, that it possesses a pure pNE. It remains to understand the computational complexity of maximizing the potential function (and thus, computing a pNE).

A promising line for future research is the consideration of equilibria which allow players to change strategies in each period of the game, instead of being committed to a decision from the beginning of it.

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A Concavity of the Potential Function for ULSG

The canonical form in MIQP for the potential function 7 is:

$$\sum_{t=1}^{T}\sum_{p=1}^{m}[-F_t^p y_t^p - C_t^p x_t^p + a_t q_t^p] - \frac{1}{2}q^\intercal Q q$$

where:

$$Q = \begin{pmatrix}
2b_1 & b_1 & b_1 & \dots & b_1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
b_1 & 2b_1 & b_1 & \dots & b_1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\
b_1 & b_1 & b_1 & \dots & 2b_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 2b_2 & b_2 & b_2 & \dots & b_2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & b_2 & 2b_2 & b_2 & \dots & b_2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & \dots & 0 & b_2 & b_2 & b_2 & \dots & 2b_2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 2b_T & b_T & b_T & \dots & b_T \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & b_T & 2b_T & b_T & \dots & b_T \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & b_T & b_T & b_T & \dots & 2b_T
\end{pmatrix}$$

and:

$$q = \begin{pmatrix} q_1^1 & q_1^2 & \dots & q_1^m & q_2^1 & q_2^2 & \dots & q_2^m & \dots & q_T^1 & q_T^2 & \dots & q_T^m \end{pmatrix}.$$

If the matrix Q is positive semi-definite, then the problem of maximizing the potential function 7's continuous relaxation over X becomes concave (concave quadratic programming optimizations can be solved in polynomial time). If the eigenvalues of Q are all positive, then Q is positive definite (in particular, semi-definite). Matrix Q is a *block matrix*, thus the eigenvalues of Q are the eigenvalues of each of its blocks; see Anton and Rorres [1] for details in linear algebra. The eigenvalues for each of the diagonal blocks of Q are given in the following lemma.

Lemma 2 *A matrix with dimension $m \times m$ and the form:*

$$B = \begin{pmatrix} 2b & b & \dots & b \\ b & 2b & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & 2b \end{pmatrix}$$

has exactly two distinct eigenvalues: $(m+1)b$ and b.

Proof. Suppose that $(x_1, x_2, \dots, x_m)$ is an eigenvector for B corresponding to an eigenvalue λ. Then by definition:

$$\begin{pmatrix} 2b & b & b & \dots & b \\ b & 2b & b & \dots & b \\ \vdots & \dots & \ddots & \dots & \vdots \\ b & b & b & \dots & 2b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \Leftrightarrow \begin{pmatrix} bx_1 + bx_2 + \dots bx_m \\ bx_1 + bx_2 + \dots bx_m \\ \vdots \\ bx_1 + bx_2 + \dots bx_m \end{pmatrix} = \begin{pmatrix} x_1(\lambda - b) \\ x_2(\lambda - b) \\ \vdots \\ x_m(\lambda - b) \end{pmatrix}.$$

One solution for the system above is the eigenspace associated with the eigenvalue b:

$$\mathcal{E}_b = \{(x_1, x_2, \dots, x_m) : x_1 + x_2 + \dots + x_m = 0\},$$

which has dimension $m - 1$ (number of linear independent vectors). Another solution is the eigenspace associated with the eigenvalue $(m + 1)b$:

$$\mathcal{E}_{(m+1)b} = \{(x_1, x_2, \dots, x_m) : x_1 = x_2 = \dots = x_m\},$$

which has dimension 1.

Note that $\mathcal{E}_b \cap \mathcal{E}_{(m+1)b} = \{(0, 0, \dots, 0)\}$, and thus the dimension of $\mathcal{E}_b \cup \mathcal{E}_{(m+1)b}$ is m, which cannot exceed the dimension of B. Therefore, $(m + 1)b$ and b are all distinct eigenvalues. □

Corollary 1 *For an ULSG with m players, the eigenvalues associated with Q are:*

$$\{(m + 1)b_1, (m + 1)b_2, \dots, (m + 1)b_T, b_1, b_2, \dots, b_T\}.$$

Corollary 2 *For an ULSG with m players, the associated Q is symmetric positive definite.*

Proof. All eigenvalues of Q are positive, since $b_t > 0$ for $t = 1, \dots, T$. □

Corollary 3 *Maximizing function 7 over the set of feasible strategies X is a concave MIQP.*